

University of Groningen

A Simultaneous Balanced Truncation Approach to Model Reduction of Switched Linear Systems

Monshizadeh, Nima; Trentelman, Hendrikus; Camlibel, M. Kanat

Published in:
IEEE Transactions on Automatic Control

DOI:
[10.1109/TAC.2012.2202031](https://doi.org/10.1109/TAC.2012.2202031)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2012

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Monshizadeh, N., Trentelman, H., & Camlibel, M. K. (2012). A Simultaneous Balanced Truncation Approach to Model Reduction of Switched Linear Systems. *IEEE Transactions on Automatic Control*, 57(12), 3118-3131. <https://doi.org/10.1109/TAC.2012.2202031>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

A Simultaneous Balanced Truncation Approach to Model Reduction of Switched Linear Systems

Nima Monshizadeh, Harry L. Trentelman, *Senior Member, IEEE*, and M. Kanat Camlibel, *Member, IEEE*

Abstract—This paper deals with model reduction by balanced truncation of switched linear systems (SLS). We consider switched linear systems whose dynamics, depending on the switching signal, switches between finitely many linear systems with a common state space. These linear systems are called the modes of the SLS. The idea is to seek for conditions under which there exists a single state space transformation that brings all modes of the SLS in balanced coordinates. As a measure of reachability and observability of the state components of the SLS, we take the average of the diagonal gramians. We then perform balanced truncation by discarding the state components corresponding to the smallest diagonal elements of this average balanced gramian. In order to carry out this program, we derive necessary and sufficient conditions under which a finite collection of linear systems with common state space can be balanced by a single state space transformation. Among other things, we derive sufficient conditions under which global uniform exponential stability of the SLS is preserved under simultaneous balanced truncation. Likewise, we derive conditions for preservation of positive realness or bounded realness of the SLS. Finally, in case that the conditions for simultaneous balancing do not hold, or we simply do not want to check these conditions, we propose to compute a suitable state space transformation on the basis of minimization of an overall cost function associated with the modes of the SLS. We show that in case our conditions do hold, this transformation is in fact simultaneously balancing, bringing us back to the original method described in this paper.

Index Terms—Model reduction, simultaneous balanced truncation, switched linear systems.

I. INTRODUCTION

SEEKING for simpler descriptions of highly complex or large-scale systems has resulted in the development of many different model reduction methods and techniques. Models of lower complexity provide a simpler description, better understanding, and easier analysis of the system. In addition, models of lower complexity make computer simulations

more tractable, and simplify controller design. Two important issues in model reduction are to obtain good error bounds, and to preserve important system properties like stability, positive realness, or bounded realness. To achieve this, extensive research, and many approaches are reported in the literature on model reduction of linear time-invariant finite-dimensional systems (for an extensive overview of the literature, see [1]). One of the most well-known techniques is model reduction by balanced truncation, first introduced in [18], and later appearing in the control system literature in [17] and [21]. In this approach, first the system is transformed into a balanced form, and next a reduced order model is obtained by truncation. There are other types of balancing approaches available in the literature. Instances of those are stochastic and positive real balancing, proposed in [3], and bounded real balancing, proposed in [20]. In addition, frequency weighted balancing has been developed to approximate the system over a range of frequencies. In [5], [16], [25] and [27], different schemes are proposed for frequency weighted balancing. Another category of model reduction approaches is formed by Krylov-based methods, based on moment matching. Among the pioneering work in this direction, we refer to [6], [8], and [12].

Despite the considerable research effort on model reduction for ordinary linear systems, developing methods for model reduction of systems with switching dynamics has only been studied in very few papers up to now (see, e.g., [2], [7], [24]).

A switched linear system, typically, involves switching between a number of linear systems, called the *modes* of the switched linear system. Hence, to apply balanced truncation techniques to a switched linear system, we may seek for a basis of the (common) state space such that the corresponding modes are all in balanced form. A natural question that arises here is under what conditions such a basis exist. In this paper, necessary and sufficient conditions are derived for the existence of such basis. The results obtained are not limited to a certain type of balancing, but are applicable to different types such as Lyapunov, bounded real, and positive real balancing.

It may happen that some state components of the switched linear system are difficult to reach and observe in some of the modes yet easy to reach and observe in others. In that case, deciding how to truncate the state variables and obtain a reduced order model is not trivial. A solution to this problem is proposed in this paper. By averaging the diagonal gramians of the individual modes in the balanced coordinates, a new diagonal matrix is obtained. This *average balanced gramian* can be used to obtain a reduced order model. In the case of Lyapunov balancing, the average balanced gramian assigns an overall degree of reachability and observability to each state component. In

Manuscript received June 29, 2011; revised March 13, 2012; accepted May 04, 2012. Date of publication June 01, 2012; date of current version November 21, 2012. Recommended by Associate Editor J. H. Braslavsky.

N. Monshizadeh and H. L. Trentelman are with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AV Groningen, The Netherlands (e-mail: n.monshizadeh@rug.nl; h.l.trentelman@math.rug.nl).

M. K. Camlibel is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AV Groningen, The Netherlands, and also with the Department of Electronics Engineering, Dogus University, 34722 Kadikoy, Istanbul, Turkey (e-mail: m.k.camlibel@rug.nl).

Color versions of one or more of the figures in this paper are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TAC.2012.2202031

TABLE I
COMMON TYPES OF BALANCING

Type	Equations
Lyapunov	$AP + PA^\top + BB^\top = 0$ $A^\top Q + QA + C^\top C = 0$
Bounded Real	$AP + PA^\top + BB^\top + (PC^\top + BD^\top)(I - DD^\top)^{-1}(PC^\top + BD^\top)^\top = 0$ $A^\top Q + QA + C^\top C + (QB + C^\top D)(I - D^\top D)^{-1}(QB + C^\top D)^\top = 0$
Positive Real	$AP + PA^\top + (PC^\top - B)(D + D^\top)^{-1}(PC^\top - B)^\top = 0$ $A^\top Q + QA + (QB - C^\top)(D^\top + D)^{-1}(QB - C^\top)^\top = 0$

this way, one can decide which state components should be discarded in order to obtain a reduced order switched linear system.

Another interesting issue is, in case that the original switched linear system is stable, how to ensure that the reduced order switched linear system retains this stability. It is well-known that the existence of a common quadratic Lyapunov function (CQLF) is a sufficient condition for global uniform exponential stability of the switched linear system, see [15]. In this paper, we will establish conditions under which the reduced order switched linear system inherits a CQLF from the original switched linear system, thus preserving global uniform exponential stability.

Similarly, one may be interested in preserving positive realness or bounded realness of the original switched linear system in the reduced order model after applying simultaneous positive real or bounded real balanced truncation. Conditions for preserving these properties are proposed in this paper. The proposed conditions are similar to those obtained for stability preserving yet with a different interpretation.

It turns out that, the higher the number of modes of the original switched linear system, the more restrictive the proposed conditions are. Therefore, in addition to simultaneous balanced truncation, we develop a more general balanced truncation-like scheme as well. Independent of the number of modes of the original switched linear system, preserving desired properties like stability, positive realness or bounded realness is then guaranteed upon the satisfaction of a single condition.

This paper is organized as follows. In Section II, some preliminaries and basic material needed in this paper are discussed. Section III is devoted to characterizing *all* balancing transformations for a single linear system. It is shown how to obtain all balancing transformations for a given pair of gramians, starting from a single transformation that diagonalizes the product of these gramians. In Section IV, we solve the problem of simultaneous balancing, and give necessary and sufficient conditions for the existence of a single balancing transformation for multiple pairs of gramians. In Section V, we apply our results on simultaneous balancing to model reduction by balanced truncation of switched linear systems. We also address the issues of preservation of stability, positive realness, and bounded realness. Since the conditions for the existence of simultaneous balancing transformations can be rather restrictive, in Section VI we propose a truncation method that is generally applicable, and, importantly, reduces to simultaneous balanced truncation if possible. Section VII is devoted to a numerical example to il-

lustrate the methods introduced in this paper. The paper closes with conclusions in Section VIII.

II. PRELIMINARIES

In this section, we will review some basic material on balancing and on simultaneous diagonalization. Consider the finite dimensional, linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$. Assume that the system is internally stable, i.e., the matrix A has all its eigenvalues in the open left half plane. Also assume (A, B) is controllable and (C, A) is observable. We shortly denote this system by $\mathcal{H} = (A, B, C, D)$. Basically, balancing the system \mathcal{H} means to find a nonsingular state space transformation T that diagonalizes appropriately chosen positive definite matrices P and Q in a covariant and contravariant manner, respectively. This means that P transforms to TPT^\top and Q transforms to $T^{-\top}QT^{-1}$, and the transformed matrices should be diagonal, or diagonal and equal. Important special cases of balancing are classical Lyapunov balancing, bounded real (BR) balancing, and positive real (PR) balancing. In Lyapunov balancing the matrices P and Q are the reachability and observability gramian associated with the system \mathcal{H} , while in BR and PR balancing P and Q are the minimal real symmetric solutions of a pair of algebraic Riccati equations, see Table I. Also, one could do balancing based on positive definite solutions of Lyapunov inequalities, instead of equations (see [4]), or use alternative solutions, not necessarily minimal, of the Riccati equations in Table I.

In this paper, we will study the concept of balancing without referring to a particular underlying system. Instead, we will start off with a given pair of positive definite matrices, and study how to find a suitable transformation for this pair. Thus, let P, Q be real symmetric positive definite $n \times n$ matrices. Then, the concepts of essentially balancing and balancing transformations are defined as follows.

Definition II.1: Let $T \in \mathbb{R}^{n \times n}$ be nonsingular. We call T an *essentially-balancing* transformation for (P, Q) if TPT^\top and $T^{-\top}QT^{-1}$ are diagonal. In this case, we say T *essentially balances* (P, Q) . We call T a *balancing transformation* for (P, Q) if $TPT^\top = T^{-\top}QT^{-1} = \Sigma$, where Σ is a diagonal matrix. In this case, we say T *balances* (P, Q) .

Note that essentially balancing transformations have been also referred to as *contragredient* transformations in the literature (see [14]). It is well-known, see for example [28], that for any pair of real symmetric positive definite matrices (P, Q) there exists a balancing transformation. It is clear that the diagonal elements of the matrix Σ in Definition II.1 coincide with the square roots of the eigenvalues of PQ . In the case of Lyapunov balancing, the diagonal elements of the corresponding Σ are the nonzero Hankel singular values (HSV) of the system. Similarly, in the case of bounded real and positive real balancing, the diagonal elements of Σ are the nonzero bounded real and positive real characteristic values, respectively (see [10], [19], [20], [23], [26]). Next, we will address the issue of simultaneous diagonalization of matrices.

Definition II.2: Let $M \in \mathbb{C}^{n \times n}$ be diagonalizable. Then the nonsingular matrix $V \in \mathbb{C}^{n \times n}$ is called a *diagonalizing transformation* for M if VMV^{-1} is diagonal. In this case, we say V *diagonalizes* M . Two diagonalizable matrices $X, Y \in \mathbb{C}^{n \times n}$ are said to be *simultaneously diagonalizable* if there exists a nonsingular matrix $V \in \mathbb{C}^{n \times n}$ such that VXV^{-1} and VYV^{-1} are both diagonal.

A necessary and sufficient condition for simultaneous diagonalizability of two given matrices is well-known, and is stated in the following lemma [11]:

Lemma II.3: Let $X, Y \in \mathbb{C}^{n \times n}$ be diagonalizable matrices. Then X and Y are simultaneously diagonalizable if and only if they commute, i.e., $XY = YX$.

The generalization of the above result to the case of three or more matrices is straightforward. In fact, a finite set of diagonalizable matrices is simultaneously diagonalizable if and only if each pair in the set commutes.

III. BALANCING TRANSFORMATIONS FOR A PAIR OF POSITIVE DEFINITE MATRICES

In this section, we will show how we can obtain *all* balancing transformations for a given pair of real symmetric positive definite matrices (P, Q) by using a single diagonalizing transformation of the product PQ . In the first part of this section we will deal with the (generic) case that the eigenvalues of the product PQ are all distinct. In the second part, we will extend our results to the general case of repeated eigenvalues of PQ .

A. Distinct Eigenvalue Case

Let P and Q be real symmetric positive definite matrices and assume that the eigenvalues of PQ are all distinct. Let \hat{T} be a balancing transformation for the pair (P, Q) , and denote the corresponding diagonal matrix Σ by $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, where $\sigma_i \neq \sigma_j$ ($i \neq j$). We note that the diagonal elements σ_i are not necessarily ordered in a decreasing manner. Clearly, \hat{T} satisfies

$$\hat{T}PQ\hat{T}^{-1} = \Sigma^2.$$

and hence the columns of \hat{T}^{-1} are eigenvectors of PQ corresponding to the distinct eigenvalues σ_i^2 , $i = 1, 2, \dots, n$. Since the eigenvalues of PQ are real, there exists $V \in \mathbb{R}^{n \times n}$ that diagonalizes PQ . Now, it is easy to observe that \hat{T} can be written as $\hat{T} = I_P D^{-1} V$ where $D \in \mathbb{R}^{n \times n}$ is a nonsingular diagonal

matrix and I_P is a permutation matrix. Clearly, the transformation T given by $T = D^{-1}V$ also balances (P, Q) . Therefore, we can always obtain a balancing transformation for (P, Q) by scaling a diagonalizing transformation of PQ .

The following lemma states that there is a one-to-one correspondence between diagonalizing transformations and essentially balancing transformations.

Lemma III.1: Let P and Q be real symmetric positive definite matrices. Assume that the eigenvalues of PQ are all distinct. Then, the matrix $V \in \mathbb{R}^{n \times n}$ is an essentially-balancing transformation for (P, Q) if and only if it is a diagonalizing transformation for PQ .

Proof: First, assume that V is an essentially-balancing transformation for (P, Q) . Then, by definition, VPV^\top and $V^{-\top}QV^{-1}$ are diagonal. Consequently, the product $VPV^\top V^{-\top}QV^{-1} = VPQV^{-1}$ is diagonal. Hence, V is a diagonalizing transformation for PQ .

Conversely, suppose V is a diagonalizing transformation for PQ . Then, by using the distinct eigenvalue assumption, it follows from the introduction to this subsection that there exists a nonsingular diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $T = D^{-1}V$ balances (P, Q) . Hence, by Definition II.1, we have

$$D^{-1}VPV^\top D^{-1} = \Sigma = DV^{-\top}QV^{-1}D.$$

Consequently, VPV^\top and $V^{-\top}QV^{-1}$ are diagonal matrices, and V is, therefore, an essentially balancing transformation for (P, Q) . ■

Based on the previous discussion and Lemma III.1, the following theorem characterizes balancing transformations for a given pair of positive definite matrices (P, Q) .

Theorem III.2: Let P and Q be real symmetric positive definite matrices. Assume the eigenvalues of PQ are all distinct. Let $V \in \mathbb{R}^{n \times n}$ be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2 . Then T is a balancing transformation for (P, Q) with corresponding diagonal matrix Σ if and only if $T = D^{-1}V$ where $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix satisfying one, and, hence, all of the following equivalent equalities:

$$D^4 = (VPV^\top)(VQ^{-1}V^\top), \quad (2)$$

$$D^2 = (VPV^\top)\Sigma^{-1} \quad (3)$$

$$D^2 = \Sigma(VQ^{-1}V^\top). \quad (4)$$

Proof: First, we show that (2), (3), and (4) are equivalent. Clearly, we have

$$VPV^\top V^{-\top}QV^{-1} = \Sigma^2. \quad (5)$$

By Lemma III.1, V is an essentially balancing transformation for (P, Q) ; therefore, VPV^\top and $V^{-\top}QV^{-1}$ are diagonal. Hence, (5) yields $(VPV^\top)\Sigma^{-1} = \Sigma VQ^{-1}V^\top$. Consequently, (3), and (4) are equivalent. In addition, their product results in (2). So, it remains to show that (2) implies (3) or (4). By (5), (2) can be rewritten as $D^4 = \Sigma^2(VQ^{-1}V^\top)^2$. Hence, $D^2 = \Sigma VQ^{-1}V^\top$. The last implication is due to the fact that a positive definite matrix has a unique positive definite square root.

Now, let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix satisfying (2). By (2), we have

$$D^{-2}VPV^\top = D^2V^{-\top}QV^{-1}. \quad (6)$$

Since VPV^\top and $V^{-\top}QV^{-1}$ are diagonal, (6) can be written as

$$D^{-1}VPV^\top D^{-1} = DV^{-\top}QV^{-1}D. \quad (7)$$

Hence, $T = D^{-1}V$ is a balancing transformation for (P, Q) . Clearly, the corresponding diagonal matrix is Σ .

To prove the converse, let T be a balancing transformation for (P, Q) with corresponding diagonal matrix Σ . Clearly, the columns of T^{-1} and V^{-1} are eigenvectors of PQ . Hence, T can be written as $T = D^{-1}V$ for some nonsingular diagonal matrix $D \in \mathbb{R}^{n \times n}$. By Definition II.1, we have

$$D^{-1}VPV^\top D^{-1} = DV^{-\top}QV^{-1}D. \quad (8)$$

By Lemma III.1, VPV^\top and $V^{-\top}QV^{-1}$ are diagonal. Therefore, by changing the order of multiplications in (8) we obtain

$$D^{-2}VPV^\top = D^2V^{-\top}QV^{-1} \quad (9)$$

which simplifies to (2). ■

The above theorem provides a straightforward method for computing balancing transformations. Given real symmetric positive definite matrices P and Q , we compute, first, a diagonalizing transformation for PQ . Obviously, this transformation can be obtained directly from the eigenvectors of PQ . Then, as stated in the theorem, a balancing transformation can be obtained by scaling the diagonalizing transformation, and the scaling matrix can be taken as any nonsingular real diagonal matrix satisfying (2), (3), or (4). Note that computation of D is simple and merely requires the multiplication of two diagonal matrices.

B. Case of Possibly Repeated Eigenvalues

We will now deal with the general case that PQ may have repeated eigenvalues. The following lemma is crucial for the proof of the subsequent results.

Lemma III.3: Let V be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2 . Let T be a balancing transformation for (P, Q) with corresponding diagonal matrix Σ . Then, Σ commutes with VT^{-1} , VPV^\top , and $VQ^{-1}V^\top$.

Proof: Clearly, we have $TPQT^{-1} = \Sigma^2$ and $VPQV^{-1} = \Sigma^2$. Hence, $VT^{-1}\Sigma^2 = \Sigma^2VT^{-1}$. Therefore, VT^{-1} commutes with Σ^2 . Thus, VT^{-1} and Σ also commute. In addition, we have

$$TPT^\top = \Sigma = T^{-\top}QT^{-1}.$$

Hence, $TV^{-1}VPV^\top V^{-\top}T^\top = \Sigma$. Thus, $VPV^\top = VT^{-1}\Sigma(VT^{-1})^\top$. Since Σ and VT^{-1} commute, we obtain $\Sigma^{-1}VPV^\top = VT^{-1}(VT^{-1})^\top$. Hence, $\Sigma^{-1}VPV^\top$ is symmetric, and Σ commutes with VPV^\top . In a similar fashion, we obtain $\Sigma VQ^{-1}V^\top = VT^{-1}(VT^{-1})^\top$. Consequently, Σ and $VQ^{-1}V^\top$ commute. ■

The following theorem now provides an extension to the repeated eigenvalue case of Theorem III.2:

Theorem III.4: Let P and Q be real symmetric positive definite matrices. Let $V \in \mathbb{R}^{n \times n}$ be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2 . Then T is a balancing transformation for (P, Q) with corresponding diagonal matrix Σ if and only if $T = \Delta^{-1}V$ where $\Delta \in \mathbb{R}^{n \times n}$ is a nonsingular matrix which commutes with Σ , and satisfies one, and, hence, all of the following equivalent equalities:

$$(\Delta\Delta^\top)^2 = (VPV^\top)(VQ^{-1}V^\top) \quad (10)$$

$$\Delta\Delta^\top = (VPV^\top)\Sigma^{-1} \quad (11)$$

$$\Delta\Delta^\top = \Sigma(VQ^{-1}V^\top) \quad (12)$$

Proof: First, we show that (10), (11), and (12) are equivalent. Clearly, we have

$$VPV^\top V^{-\top}QV^{-1} = \Sigma^2. \quad (13)$$

By Lemma III.3, we obtain $(VPV^\top)\Sigma^{-1} = \Sigma VQ^{-1}V^\top$. Hence, (11), and (12) are equivalent. In addition, the product of (11) and (12) results in (10). So, it remains to show (10) implies (11) or (12). By (13), we can rewrite (10) as $(\Delta\Delta^\top)^2 = \Sigma^2(VQ^{-1}V^\top)^2$. Hence, we obtain $\Delta\Delta^\top = \Sigma VQ^{-1}V^\top$. Note that $\Delta\Delta^\top$ and $\Sigma VQ^{-1}V^\top$ are positive definite matrices.

Now, assume T is a balancing transformation for (P, Q) . Clearly, we can write T as $T = \Delta^{-1}V$ for a nonsingular matrix Δ . Hence,

$$\Delta^{-1}VPV^\top\Delta^{-\top} = \Sigma$$

which simplifies to (11) by Lemma III.3.

To prove the converse, assume that the nonsingular matrix Δ satisfies $\Sigma\Delta = \Delta\Sigma$, (11), and, equivalently, (12). Clearly, Δ^\top , Δ^{-1} , and $\Delta^{-\top}$ also commute with Σ . Hence, we have

$$\begin{aligned} \Delta\Sigma\Delta^\top &= VPV^\top \\ \Delta^{-\top}\Sigma\Delta^{-1} &= V^{-\top}QV^{-1} \end{aligned}$$

Now, it is easy to observe that the transformation T given by $T = \Delta^{-1}V$, balances (P, Q) with corresponding diagonal matrix Σ . ■

The above theorem is valid for any choice of diagonalizing transformation V , where repeated eigenvalues do not necessarily have to be grouped together on the diagonal of Σ . We will apply the statement of this theorem in this form in our subsequent results on simultaneous balancing. If, however, we would restrict ourselves to diagonalizing transformations V that do group together repeated eigenvalues on the diagonal of Σ , we obtain the following more specific result:

Corollary III.5: Let P and Q be real symmetric positive definite matrices. Let $V \in \mathbb{R}^{n \times n}$ be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2 such that

$$\Sigma = \begin{pmatrix} \sigma_1 I_{m_1} & 0 & \dots & 0 \\ 0 & \sigma_2 I_{m_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N I_{m_N} \end{pmatrix} \quad (14)$$

with $\sigma_i \neq \sigma_j$ for $i \neq j$, and where m_i is the algebraic multiplicity of σ_i , and I_{m_i} is the identity matrix of size m_i , $i = 1, 2, \dots, N$. Then T is a balancing transformation for (P, Q) with corresponding diagonal matrix Σ if and only if $T = \Delta^{-1}V$ where $\Delta \in \mathbb{R}^{n \times n}$ is a nonsingular block-diagonal matrix with N square blocks of sizes m_1, m_2, \dots, m_N , respectively, and Δ satisfies one, and, hence, all of the equivalent equalities (10), (11), and (12).

Proof: Obviously, an $n \times n$ matrix Δ commutes with the diagonal matrix Σ in (14) if and only if Δ is blockdiagonal with diagonal blocks of size $m_i \times m_i$. Therefore, the result of Theorem III.4 simplifies to Corollary III.5. ■

IV. SIMULTANEOUS BALANCING FOR MULTIPLE PAIRS OF POSITIVE DEFINITE MATRICES

In this section, we will study the question under what conditions multiple pairs of real symmetric positive definite matrices can be simultaneously balanced by one and the same transformation. We will start off by considering the problem for two pairs of positive definite matrices (P_1, Q_1) and (P_2, Q_2) .

Before stating and proving the main result of this section, we give the following instrumental lemma:

Lemma IV.1: Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric positive definite matrix. Let LL^T be a Cholesky decomposition of M , that is, $M = LL^T$ where $L \in \mathbb{R}^{n \times n}$ is a lower triangular matrix with positive diagonal entries. Then, M commutes with Σ if and only if L commutes with Σ .

Proof: The proof of the “if” part is trivial. To prove the “only if” part, assume that M commutes with Σ . Clearly, we have $LL^T \Sigma = \Sigma LL^T$. Hence,

$$L^T \Sigma L^{-T} = L^{-1} \Sigma L. \quad (15)$$

Now, since the left-hand side of (15) is upper triangular and the right hand side is lower triangular, they should be equal to a diagonal matrix. Clearly, this diagonal matrix must be equal to Σ . Hence, we have $L^{-1} \Sigma L = \Sigma$ which yields $\Sigma L = L \Sigma$. ■

The following theorem gives necessary and sufficient conditions for the existence of a transformation T that simultaneously balances (P_1, Q_1) and (P_2, Q_2) .

Theorem IV.2: Let (P_1, Q_1) and (P_2, Q_2) be pairs of real symmetric positive definite matrices. There exists a transformation T that balances both (P_1, Q_1) and (P_2, Q_2) if and only if the following two conditions hold:

- i) $P_1 Q_1$ and $P_2 Q_2$ commute,
- ii) $P_1 Q_2 = P_2 Q_1$.

Proof: Assume T is a balancing transformation for both (P_1, Q_1) and (P_2, Q_2) . Clearly, T is a diagonalizing transformation for both $P_1 Q_1$ and $P_2 Q_2$. Hence, by Lemma II.3, $P_1 Q_1$ and $P_2 Q_2$ commute. In addition, by Definition II.1, we have

$$\begin{aligned} T P_1 T^T &= T^{-T} Q_1 T^{-1} \\ T P_2 T^T &= T^{-T} Q_2 T^{-1}. \end{aligned} \quad (16)$$

Consequently, $T P_1 Q_2 T^{-1} = T P_2 Q_1 T^{-1}$ which yields $P_1 Q_2 = P_2 Q_1$.

Conversely, assume that the conditions i) and ii) hold. Since $P_1 Q_1$ and $P_2 Q_2$ commute, there exists a nonsingular matrix V that simultaneously diagonalizes $P_1 Q_1$ and $P_2 Q_2$ with corresponding diagonal matrices Σ_1^2 and Σ_2^2 , respectively. Clearly, we have

$$V P_1 Q_1 P_2 Q_2 V^{-1} = \Sigma_1^2 \Sigma_2^2. \quad (17)$$

By condition ii), we have $Q_2 P_1 = Q_1 P_2$. Hence, (17) can be rewritten as $V (P_1 Q_2)^2 V^{-1} = \Sigma_1^2 \Sigma_2^2$. Consequently, we have $(P_1 Q_2)^2 = V^{-1} (\Sigma_1 \Sigma_2)^2 V$. Now, since $P_1 Q_2$ has only positive eigenvalues, we obtain $P_1 Q_2 = V^{-1} \Sigma_1 \Sigma_2 V$. Therefore,

$$V P_1 V^T V^{-T} Q_2 V^{-1} = \Sigma_1 \Sigma_2$$

which, by Lemma III.3, yields

$$V P_1 V^T \Sigma_1^{-1} = \Sigma_2 V Q_2^{-1} V^T. \quad (18)$$

Since (18) is a real symmetric positive definite matrix, it admits a Cholesky decomposition

$$V P_1 V^T \Sigma_1^{-1} = \Sigma_2 V Q_2^{-1} V^T = \Delta \Delta^T$$

(where Δ is a lower triangular matrix with positive diagonal entries). Now, again by Lemma III.3, the identical terms in (18) commute with both Σ_1 and Σ_2 . Hence, by Lemma IV.1, Δ commutes with both Σ_1 and Σ_2 . Thus, by Theorem III.4, $T = \Delta^{-1}V$ simultaneously balances (P_1, Q_1) and (P_2, Q_2) . ■

For the sake of simplicity, the above result and conditions have been stated for two pairs of positive definite matrices (P_1, Q_1) and (P_2, Q_2) . The generalization of the results to k pairs of positive definite matrices, with $k \geq 2$ is straightforward, and is stated in the following corollary.

Corollary IV.3: Let $(P_1, Q_1), (P_2, Q_2), \dots, (P_k, Q_k)$ be k pairs of real symmetric positive definite matrices. There exists a transformation T that simultaneously balances $(P_1, Q_1), (P_2, Q_2), \dots, (P_k, Q_k)$ if and only if the following conditions hold:

- i) $P_i Q_i$ and $P_j Q_j$ commute for all $i, j = 1, 2, \dots, k$.
- ii) $P_i Q_j = P_j Q_i$ for all $i, j = 1, 2, \dots, k$.

It is worth mentioning that in the particular case where for each i the eigenvalues of $P_i Q_i$ are all distinct, the first condition of Corollary IV.3 by itself already implies the existence of a simultaneous essentially-balancing transformation for the pairs (P_i, Q_i) , $i = 1, 2, \dots, k$. This follows immediately from Lemma III.1.

V. MODEL REDUCTION OF SWITCHED LINEAR SYSTEMS BY SIMULTANEOUS BALANCED TRUNCATION

A. Balanced Truncation of Switched Linear Systems

We will now apply our previous results to model reduction by balanced truncation of switched linear systems. As indicated before, in the context of model reduction of linear systems, the matrices P and Q can be taken to be the solutions of the Lyapunov equations, or the minimal solutions of the bounded real or positive real Riccati equations (see Table I). Consequently, the results stated in the previous sections cover different types of balancing of a single linear system, and, moreover, indicate the possibility of simultaneous balancing for multiple linear systems. Thus, simultaneous balancing, if possible, provides a straightforward approach to model reduction of certain hybrid systems. Except from having the same state space dimension, no assumption is needed regarding the relation of the individual modes of the hybrid system.

In this section, we will first treat the P and Q matrices as reachability and observability gramians, thus applying our results to Lyapunov balanced truncation of switched linear systems.

A typical switched linear system (SLS) is described by (see [15]):

$$\begin{aligned}\dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u\end{aligned}\quad (19)$$

where σ is a piecewise constant function of time, t , taking its value from the index set $K = \{1, 2, \dots, k\}$, and $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{p \times n}$, $D_i \in \mathbb{R}^{p \times m}$ for all $i \in K$. Let $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$ denote the i th mode of the given SLS. Assume that \mathcal{H}_i is internally stable, controllable, and observable for every i . Let P_i and Q_i be the reachability and observability gramians of \mathcal{H}_i , respectively. Now, if the conditions of Corollary IV.3 hold for the pairs of gramians $(P_1, Q_1), (P_2, Q_2), \dots, (P_k, Q_k)$, then there exists a single state space transformation $T \in \mathbb{R}^{n \times n}$ that simultaneously balances all k modes of the given SLS. Consequently, by applying T to the individual modes of (19) and truncating, reduced order modes can be obtained.

As mentioned in the introduction, it may occur that some states are relatively difficult to reach and observe in some of the modes, yet easy to reach and observe in others. In order to measure the degree of reachability and observability of each of the state components of the overall SLS, we propose to take the *average over all modes* of the corresponding Hankel singular values. More precisely:

Definition V.1: Assume that simultaneous balancing of the modes of the SLS (19) is possible, and let $\Sigma_1, \Sigma_2, \dots, \Sigma_k$ be the gramians of the modes of (19) after simultaneous balancing. Then we define the *average balanced gramian*, denoted by Σ_{av} , as

$$\Sigma_{av} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + \dots + \Sigma_k). \quad (20)$$

Note that the average balanced gramian is unique up to permutation of its diagonal elements. Now, the diagonal matrix Σ_{av} indicates which states of the SLS are important and which are

negligible, and can be used to obtain a reduced order SLS. In fact, the i th diagonal element of Σ_{av} assigns an overall degree of reachability and observability to the i th state component of the balanced representation. Suppose that $\bar{\sigma}_1 > \bar{\sigma}_2 > \dots > \bar{\sigma}_N$ are the *distinct* diagonal elements of the average balanced gramian, where $\bar{\sigma}_i$ appears m_i times, $\sum_i m_i = n$. Let $1 \leq l \leq N$ be an integer. By discarding the state components corresponding to the $N - l$ smallest distinct diagonal elements of Σ_{av} we can reduce the order by $\sum_{i=l+1}^N m_i$ and obtain a reduced order SLS with state space dimension $r = \sum_{i=1}^l m_i$. This leads to a reduced-order SLS

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}_\sigma \bar{x} + \bar{B}_\sigma u \\ \bar{y} &= \bar{C}_\sigma \bar{x} + \bar{D}_\sigma u\end{aligned}\quad (21)$$

with modes $\bar{\mathcal{H}}_i = (\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i)$ ($i = 1, 2, \dots, k$), where $\bar{A}_i \in \mathbb{R}^{r \times r}$, $\bar{B}_i \in \mathbb{R}^{r \times m}$, $\bar{C}_i \in \mathbb{R}^{p \times r}$, and $\bar{D}_i = D_i$.

In the particular case that for each i the product $P_i Q_i$ has distinct eigenvalues, equivalently mode \mathcal{H}_i has distinct Hankel singular values, it is a well-known fact that the individual truncated modes $\bar{\mathcal{H}}_i$ preserve the property of internal stability of the original modes \mathcal{H}_i for all $i = 1, 2, \dots, k$. In the case of repeated Hankel singular values, internal stability of the truncated individual modes is only guaranteed if, for each mode, all state components corresponding to a repeated, to be discarded, HSV are truncated in that mode. Preservation of internal stability of the individual modes is in fact not the kind of stability preservation that we are looking for in SLS. Instead, in Section V-B we will address the issue of preservation of *global uniform exponential stability*, of the SLS.

Remark V.2: If the individual modes of the given SLS are not of equal importance, or if some information on the switching signal is available, the overall balanced gramian can be defined as a *weighted* average of the gramians of the individual modes in balanced coordinates. In this way one can take into account the importance of the different modes by adjusting the weighting coefficients.

Of course, our results on simultaneous balancing can also be applied to positive real and bounded real balancing, see [10], [20], [26]. The single linear system (1) with $p = m$ is positive real if and only if the linear matrix inequality (LMI)

$$\begin{pmatrix} A^\top K + K A & K B - C^\top \\ B^\top K - C & -D - D^\top \end{pmatrix} \leq 0 \quad (22)$$

has a real symmetric positive definite solution K . It is well known that in that case the LMI has extremal real symmetric solutions K_{\min} and K_{\max} , and $0 < K_{\min} \leq K_{\max}$. In the context of PR balancing we take $Q = K_{\min}$ and $P = K_{\max}^{-1}$, and (1) is called PR balanced if $P = Q$ is diagonal. If $D + D^\top$ is invertible, then P and Q coincide with the minimal solutions of the PR Riccati equations in Table I.

For bounded real balancing the same holds. The system (1) is bounded real if the LMI

$$\begin{pmatrix} A^\top K + K A + C^\top C & K B + C^\top D \\ B^\top K + D^\top & -I + D^\top D \end{pmatrix} \leq 0 \quad (23)$$

has a real symmetric positive definite solution K . In that case, the LMI has extremal real symmetric solutions K_{\min} and K_{\max}

satisfying $0 < K_{\min} \leq K_{\max}$. Again we take $Q = K_{\min}$ and $P = K_{\max}^{-1}$, and (1) is called BR balanced if $P = Q$ is diagonal. If $I - D^T D$ is invertible, then P and Q coincide with the minimal solutions of the BR Riccati equations in Table I.

Now, again consider the SLS (19) with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$. Similar as in the case of Lyapunov balancing, both in the PR and the BR case we denote the diagonal matrices obtained after simultaneously (PR or BR) balancing the pairs (P_i, Q_i) corresponding to the modes \mathcal{H}_i by Σ_i , and define the average balanced gramian by (20).

In Section V-C, we will review the concepts of positive realness and bounded realness of switched linear systems, and study the preservation of these properties under simultaneous PR and BR balanced truncation of the individual modes of the SLS.

Remark V.3: In the context of model reduction for a single linear system, error bounds, in the sense of L_2 gain of the error dynamics, have been extensively studied in the literature (see [1] for details). These bounds are, typically, stated in terms of the neglected HSV, BR or PR characteristic values, depending on the model reduction technique adopted. In a similar fashion to the linear case, one can take the L_2 -gain of the switched linear system associated with the error dynamics as a measure of model reduction error. The most plausible condition guaranteeing finiteness of the L_2 -gain of a given SLS with respect to arbitrary switching is to require that each individual subsystem has a finite L_2 -gain established by a common storage function (see [9, Th. 1]). Such L_2 -gain based error bounds have recently been studied in [22] under the assumption that the individual subsystems admit common (generalized) gramians. On the other hand, one could consider developing simultaneous balancing techniques in the discrete time domain, and try to establish error bounds similar to those already available in the literature (see, e.g., [2], [13]).

B. Preservation of Stability Under Arbitrary Switching

As already observed in the previous subsection, suitable simultaneous balanced truncation of the individual modes of a switched linear system yields a reduced-order switched linear system whose individual modes are internally stable. Of course, this does not mean that the SLS itself is stable. In the present subsection we will obtain conditions under which simultaneous balanced truncation preserves the stability of the SLS.

The concept of stability that we will use here is that of *global uniform exponential stability*. We call the SLS given by (19) globally uniformly exponentially stable if there exist positive constants α and β such that the solution $x(t)$ of $\dot{x} = A_\sigma x$ for any initial state $x(0)$ and any switching signal σ satisfies $\|x(t)\| \leq \beta e^{-\alpha t} \|x(0)\|$ for all $t \geq 0$ (see [15]). A sufficient condition for global uniform exponential stability of an SLS is that the state matrices of the individual modes share a common quadratic Lyapunov function (CQLF) (see [15]). That is, there exists a real symmetric positive definite matrix X such that $A_i^T X + X A_i < 0$ for all $i = 1, 2, \dots, k$. Assuming that the state matrices of the modes of the given SLS enjoy this property, we seek for conditions under which this property is preserved in the reduced order SLS. This leads us to the following theorem.

Theorem V.4: Consider the switched linear system (19) with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \dots, k$. Assume that there exists $X > 0$ such that $A_i^T X + X A_i < 0$ for all $i = 1, 2, \dots, k$. Let P_i and Q_i be the reachability and observability gramians, respectively, of the i th mode \mathcal{H}_i . Suppose that the following conditions hold:

- i) $P_i Q_i$ and $P_j Q_j$ commute for all $i, j = 1, 2, \dots, k$.
- ii) $P_i Q_j = P_j Q_i$ for all $i, j = 1, 2, \dots, k$.
- iii) $X P_i Q_i = Q_i P_i X$ for all $i = 1, 2, \dots, k$.

Then there exists a state space transformation that simultaneously balances all modes \mathcal{H}_i for $i = 1, 2, \dots, k$. Moreover, let $\bar{\sigma}_1 > \bar{\sigma}_2 > \dots > \bar{\sigma}_N$ be the *distinct* diagonal elements of the average balanced gramian, where $\bar{\sigma}_i$ appears m_i times. Then, for each positive integer $1 \leq l \leq N$, the truncated SLS of order $r = \sum_{i=1}^l m_i$ given by (21) is globally uniformly exponentially stable.

Proof: By Corollary IV.3, simultaneous balancing is possible upon satisfaction of conditions i) and ii). By condition iii) we have

$$X^{\frac{1}{2}} P_i Q_i X^{-\frac{1}{2}} = X^{-\frac{1}{2}} Q_i P_i X^{\frac{1}{2}}.$$

Hence, $X^{1/2} P_i Q_i X^{-(1/2)}$ is symmetric. In addition, condition i) implies that $X^{1/2} P_i Q_i X^{-(1/2)}$ and $X^{1/2} P_j Q_j X^{-(1/2)}$ commute for all $i, j = 1, 2, \dots, k$. Therefore, there exists an orthogonal matrix U which diagonalizes $X^{1/2} P_i Q_i X^{-(1/2)}$, for all $i = 1, 2, \dots, k$ (see [11, p. 103]). Hence, $U X^{1/2}$ is a diagonalizing transformation for $P_i Q_i$, $i = 1, 2, \dots, k$. Consequently, based on the proof of Theorem V.4, a simultaneous balancing transformation T can be obtained as $T = \Delta^{-1} U X^{1/2}$ for some nonsingular real matrix Δ . Without loss of generality, we assume that the corresponding diagonal matrix Σ_{av} given by (20) is in form of (14). Otherwise, we can multiply T by a permutation matrix from the left to achieve so. Applying T to the individual modes of the given SLS, the state matrices in the new coordinates are given by

$$\tilde{A}_i = \Delta^{-1} U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^T \Delta. \quad (24)$$

By our assumption regarding the CQLF, we have $A_i^T X + X A_i < 0$ for all $i = 1, 2, \dots, k$. Hence, for all i we have

$$\Delta^T U X^{-\frac{1}{2}} (A_i^T X + X A_i) X^{-\frac{1}{2}} U^T \Delta < 0$$

which yields

$$\Delta^T U X^{-\frac{1}{2}} A_i^T X^{\frac{1}{2}} U^T \Delta + \Delta^T U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^T \Delta < 0.$$

This can be rewritten as

$$\Delta^T U X^{-\frac{1}{2}} A_i^T X^{\frac{1}{2}} U^T \Delta^{-T} \Delta^T \Delta + \Delta^T \Delta \Delta^{-1} U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^T \Delta < 0$$

which, by (24), simplifies to

$$\tilde{A}_i^T \Delta^T \Delta + \Delta^T \Delta \tilde{A}_i < 0. \quad (25)$$

Since $T = \Delta^{-1} V$ is a simultaneous balancing transformation, by Lemma III.3 Δ commutes with Σ_i for all $i = 1, 2, \dots, k$.

Hence, Δ commutes with Σ_{av} . Therefore, Δ^\top and $\Delta^\top \Delta$ commute with Σ_{av} . Since Σ_{av} is in the form of (14), $\Delta^\top \Delta$ will have a block-diagonal structure compatible with the multiplicities m_i of the diagonal elements of Σ_{av} . Clearly, any principal submatrix of the positive definite matrix $\Delta^\top \Delta$ is positive definite. Hence, the state matrices of the reduced order SLS (21) share a CQLF. Consequently, the r th-order reduced SLS model is globally uniformly exponentially stable. ■

Note that in the above theorem, the first two conditions in Theorem V.4 implies the possibility of simultaneous balancing whereas the third condition guarantees the existence of a CQLF for the reduced order model.

C. Preservation of Passivity and Contractivity Under Arbitrary Switching

It is well known that PR or BR balanced truncation of a single linear system preserves positive realness or bounded realness. As was the case with stability, this does not mean that positive realness or bounded realness of a SLS is preserved under simultaneous balanced truncation. In the present subsection we will address this issue.

We first give a definition of dissipativity for switched linear systems. Consider the SLS (19), and let $s : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ be a given function. This function is called the *supply rate*.

Definition V.5: We call the SLS (19) dissipative with respect to supply rate $s(u, y)$ if there exists a nonnegative function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the dissipation inequality

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t)) dt \quad (26)$$

holds for all $t_0 \leq t_1$, for all switching signals σ and trajectories (x, u, y) which satisfy the system equations. Any nonnegative function V that satisfies (V.5) is called a *storage function*.

Then, we call the SLS (19) positive real if $p = m$ and if it is dissipative with respect to the supply rate $s = u^\top y$. Similarly, we call the SLS (19) bounded real if it is dissipative with respect to the supply rate $s = u^\top u - y^\top y$. It is easy to observe that if the individual modes of the SLS possess a common quadratic storage function (CQSF) for a given supply rate s , then the overall SLS is dissipative with respect to s . By applying this fact to the special cases of positive realness and bounded realness, we find that SLS (19) is positive real if there exists a real symmetric positive definite matrix K such that the following LMI's hold for all $i = 1, 2, \dots, k$:

$$\begin{pmatrix} A_i^\top K + KA_i & KB_i - C_i^\top \\ B_i^\top K - C_i & -D_i - D_i^\top \end{pmatrix} \leq 0. \quad (27)$$

Similarly, the SLS (19) is bounded real if there exists a real symmetric positive definite matrix K such that

$$\begin{pmatrix} A_i^\top K + KA_i + C_i^\top C_i & KB_i + C_i^\top D_i \\ B_i^\top K + D_i^\top C_i & -I + D_i^\top D_i \end{pmatrix} \leq 0 \quad (28)$$

holds for all $i = 1, 2, \dots, k$. In both cases, (27) and (28), we denote by $K_{i,\min}$ and $K_{i,\max}$ the extremal real symmetric solutions to the i th LMI.

Assuming the original SLS is PR or BR, it is desirable to preserve these properties in the reduced order SLS. The following theorem deals with this issue:

Theorem V.6: Consider the switched linear system (19) with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \dots, k$. Assume there exists $K > 0$ such that the LMI (27) (the LMI (28)) holds for all $i = 1, 2, \dots, k$. Define $Q_i := K_{i,\min}$ and $P_i := K_{i,\max}^{-1}$. Suppose that the following conditions hold:

- i) $P_i Q_i$ and $P_j Q_j$ commute for all $i, j = 1, 2, \dots, k$.
- ii) $P_i Q_j = P_j Q_i$ for all $i, j = 1, 2, \dots, k$.
- iii) $K P_i Q_i = Q_i P_i K$ for all $i = 1, 2, \dots, k$.

Then there exists a state space transformation that simultaneously PR balances (BR balances) all modes \mathcal{H}_i for $i = 1, 2, \dots, k$. Moreover, let $\bar{\sigma}_1 > \bar{\sigma}_2 > \dots > \bar{\sigma}_N$ be the distinct diagonal elements of the average balanced gramian Σ_{av} , where $\bar{\sigma}_i$ appears m_i times. Then, for each positive integer $1 \leq l \leq N$, the truncated SLS of order $r = \sum_{i=1}^l m_i$ given by (21) is positive real (bounded real).

Proof: We only give the proof for the PR case, the BR case is similar. By Corollary IV.3, simultaneous balancing is possible if the first two conditions hold. Using the third condition, and following a similar argument as in the proof of Theorem V.4, a simultaneous balancing transformation is given by $T = \Delta^{-1} U K^{1/2}$ for some nonsingular real matrix Δ and an orthogonal matrix U . Again, without loss of generality, we assume that the corresponding diagonal matrix Σ_{av} , given by (20), is in the form (14). Applying T to the individual modes of the given SLS, the state space representations of the modes in the new (balanced) coordinates are given by

$$\tilde{\mathcal{H}}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i)$$

where

$$\begin{aligned} \tilde{A}_i &= \Delta^{-1} U K^{\frac{1}{2}} A_i K^{-\frac{1}{2}} U^\top \Delta \\ \tilde{B}_i &= \Delta^{-1} U K^{\frac{1}{2}} B_i \\ \tilde{C}_i &= C_i K^{-\frac{1}{2}} U^\top \Delta, \quad \tilde{D}_i = D_i \end{aligned} \quad (29)$$

for $i = 1, 2, \dots, k$. By assumption, there exists a single positive definite matrix K such that for all $i = 1, 2, \dots, k$

$$\begin{pmatrix} A_i^\top K + KA_i & KB_i - C_i^\top \\ B_i^\top K - C_i & -D_i - D_i^\top \end{pmatrix} \leq 0. \quad (30)$$

Hence,

$$\begin{pmatrix} \Delta^\top U K^{-\frac{1}{2}} & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A_i^\top K + KA_i & KB_i - C_i^\top \\ B_i^\top K - C_i & -D_i - D_i^\top \end{pmatrix} \times \begin{pmatrix} K^{-\frac{1}{2}} U^\top \Delta & 0 \\ 0 & I_m \end{pmatrix} \leq 0 \quad (31)$$

for all $i = 1, 2, \dots, k$.

Using (29), the inequality (31) simplifies to

$$\begin{pmatrix} \tilde{A}_i^\top \Delta^\top \Delta + \Delta^\top \Delta A_i & \Delta^\top \Delta \tilde{B}_i - \tilde{C}_i^\top \\ \tilde{B}_i^\top \Delta^\top \Delta - \tilde{C}_i & -\tilde{D}_i - \tilde{D}_i^\top \end{pmatrix} \leq 0. \quad (32)$$

Let $\bar{A}_i \in \mathbb{R}^{r \times r}$, $\bar{B}_i \in \mathbb{R}^{r \times m}$, $\bar{C}_i \in \mathbb{R}^{m \times r}$, $\bar{D}_i \in \mathbb{R}^{m \times m}$, and $\bar{\Delta} \in \mathbb{R}^{r \times r}$ be the matrices obtained by truncating \tilde{A}_i , \tilde{B}_i , \tilde{C}_i , \tilde{D}_i , and Δ , respectively. Since this truncation is done based on Σ_{av} , $\Delta^\top \Delta$ has a block diagonal structure with respect to the

multiplicities of the diagonal elements of Σ_{av} (see the proof of Theorem V.4). Hence, the modes of the reduced order SLS, $\tilde{H}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i)$, satisfy

$$\begin{pmatrix} \tilde{A}_i^\top \tilde{\Delta}^\top \tilde{\Delta} + \tilde{\Delta}^\top \tilde{\Delta} \tilde{A}_i & \tilde{\Delta}^\top \tilde{\Delta} \tilde{B}_i - \tilde{C}_i^\top \\ \tilde{B}_i^\top \tilde{\Delta}^\top \tilde{\Delta} - \tilde{C}_i & -\tilde{D}_i - \tilde{D}_i^\top \end{pmatrix} \leq 0 \quad (33)$$

for all $i = 1, 2, \dots, k$, due to the fact that (33) is a principal submatrix of (32). Therefore, the r th order reduced SLS model is positive real. ■

Remark V.7: For each i , let P_i and Q_i be defined as in Theorems V.4 (Theorem V.6). It can be proven that if there exists at least one $\ell \in \{1, 2, \dots, k\}$ such that $P_\ell Q_\ell$ has distinct eigenvalues, then the k constraints involved in the third condition of Theorem V.4 (Theorem V.6) can be replaced by the single constraint $X P_\ell Q_\ell = Q_\ell P_\ell X$ ($K P_\ell Q_\ell = Q_\ell P_\ell K$). In fact, this new condition guarantees that the gramian X (the gramian K) will be diagonal in the balanced coordinates, which suffices to ensure preservation of GUES (positive/bounded realness).

VI. MODEL REDUCTION OF SLS BY MINIMIZING AN OVERALL COST FUNCTION

As discussed in the previous section, if certain conditions are satisfied, the technique of simultaneous balanced truncation can be applied to switched linear systems (see Theorems V.4 and V.6). Although conceptually attractive as a model reduction method, in general the conditions are rather restrictive. In many cases, like electrical circuits with certain switching topologies, the state matrices of the different modes are not completely arbitrary, and satisfy certain conditions to comply with Kirchhoff's laws. This may decrease the restrictiveness of the obtained conditions in practice.

On the other hand, the number of conditions stated in Theorem V.4 depends on the number of modes of the original SLS. Clearly, increasing the number of modes makes it harder for all conditions to be satisfied. This motivates us to propose a more general model reduction approach for the case where simultaneous balancing cannot be achieved.

It is well-known that the problem of finding a balancing transformation for a single linear system can be given a variational interpretation, and can be formulated as finding a nonsingular matrix T such that the following cost function is minimized (see [1]):

$$f(T) = \text{trace}[TPT^\top + T^{-\top}QT^{-1}] \quad (34)$$

In a SLS with k modes, thus, we deal with minimizing the following k cost functions with respect to T :

$$f_i(T) = \text{trace}[TP_iT^\top + T^{-\top}Q_iT^{-1}] \quad (35)$$

Clearly, if and only if the conditions of Corollary IV.3 hold, simultaneous balancing is possible, and there exists a transformation T which simultaneously minimizes f_i for all $i = 1, 2, \dots, k$. Otherwise, simultaneous balancing is not possible, and we should seek for a single transformation that makes the above k cost functions simultaneously as small as possible. For this, we propose to introduce a single overall cost function. Since the essence of "trace" is summation, a natural choice for this overall cost function is the sum or, equivalently, the average

of the cost functions of the individual modes. Hence, we define an overall cost function f_{av} as

$$f_{av}(\tilde{T}) = \frac{1}{k} \sum_{i=1}^k \text{trace}[\tilde{T}P_i\tilde{T}^\top + \tilde{T}^{-\top}Q_i\tilde{T}^{-1}] \quad (36)$$

It is interesting to note that in the case of balancing for a single linear system, we seek for a basis so that the sum of the eigenvalues of the positive definite matrices P and Q takes its minimum value [see (34)], while in the case of balancing for SLS we seek for a basis in which the sum of the sum of the eigenvalues of P_i and Q_i over all modes is minimal [see (36)]. Hence, minimizing the proposed overall cost function provides a natural extension of classical balancing to the case of SLS.

The cost function (36) can be restated as

$$f_{av}(\tilde{T}) = \text{trace}[\tilde{T}P_{av}\tilde{T}^\top + \tilde{T}^{-\top}Q_{av}\tilde{T}^{-1}] \quad (37)$$

where

$$P_{av} = \frac{1}{k} \sum_{i=1}^k P_i \quad \text{and} \quad Q_{av} = \frac{1}{k} \sum_{i=1}^k Q_i. \quad (38)$$

Therefore, the transformation \tilde{T} which minimizes the proposed overall cost function is exactly the one which balances the pair (P_{av}, Q_{av}) of average gramians. Consequently, \tilde{T} can be conveniently computed by the use of Corollary III.5 with respect to (P_{av}, Q_{av}) .

By applying \tilde{T} to the individual modes and truncating, a reduced order model can be obtained. Clearly, after the transformation \tilde{T} , the new state space descriptions of the individual modes are not necessarily balanced, but are expected to be relatively close to being balanced. It is of course desirable that in case where simultaneous balancing is possible, minimizing the proposed cost function yields a transformation that simultaneously balances all modes. This important issue is addressed in the following Proposition:

Proposition VI.1: Let $(P_1, Q_1), (P_2, Q_2), \dots, (P_k, Q_k)$ be k pairs of positive definite matrices. Assume that there exists a simultaneous balancing transformation for these k pairs. Assume that the product $P_{av}Q_{av}$ of the average gramians (38) has all distinct eigenvalues. Then any balancing transformation \tilde{T} for (P_{av}, Q_{av}) , with corresponding diagonal matrix $\tilde{\Sigma}$, simultaneously balances (P_i, Q_i) for all $i = 1, 2, \dots, k$. Moreover,

$$\tilde{\Sigma} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + \dots + \Sigma_k) \quad (39)$$

where Σ_i is the corresponding diagonal matrix obtained after applying \tilde{T} to (P_i, Q_i) .

Proof: Let T be a simultaneously balancing transformation for (P_i, Q_i) , $i = 1, 2, \dots, k$. Clearly, we have

$$TP_iT^\top = T^{-\top}Q_iT^{-1} = \Gamma_i \quad (40)$$

for certain diagonal matrices Γ_i , $i = 1, 2, \dots, k$. Hence,

$$TP_{av}T^\top = T^{-\top}Q_{av}T^{-1} = \frac{1}{k}(\Gamma_1 + \Gamma_2 + \dots + \Gamma_k).$$

Consequently, the transformation T balances (P_{av}, Q_{av}) . Obviously, T is also a diagonalizing transformation for $P_{av}Q_{av}$.

Thus, by Theorem III.2, it is easily verified that \tilde{T} can be written as $\tilde{T} = I_P S T$ for some permutation matrix I_P and a sign matrix S (note that, in fact, the diagonal matrix D in (2) of Theorem III.2 satisfies $D^4 = I$; hence, D is obtained as a sign matrix in this case). Now, multiplying (40) from the left by $I_P S$ and from the right by $S I_P^\top$, we obtain

$$\tilde{T} P_i \tilde{T}^\top = \tilde{T}^{-\top} Q_i \tilde{T}^{-1} = \Sigma_i \quad (41)$$

where $\Sigma_i = I_P \Gamma_i I_P^\top$, $i = 1, 2, \dots, k$. Therefore, \tilde{T} simultaneously balances (P_i, Q_i) for all $i = 1, 2, \dots, k$. By (41), we obtain

$$\tilde{T} P_{av} \tilde{T}^\top = \tilde{T}^{-\top} Q_{av} \tilde{T}^{-1} = \frac{1}{k} (\Sigma_1 + \Sigma_2 + \dots + \Sigma_k).$$

Therefore, $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma} = \frac{1}{k} (\Sigma_1 + \Sigma_2 + \dots + \Sigma_k). \quad \blacksquare$$

By Proposition VI.1, balancing and truncation based on the average gramians contains the simultaneous balancing problem as a special case. In fact, in case that simultaneous balancing is possible, the transformation which balances (P_{av}, Q_{av}) also balances (P_i, Q_i) for all $i = 1, 2, \dots, k$. Moreover in that case, the diagonal matrix $\tilde{\Sigma}$ obtained by balancing (P_{av}, Q_{av}) is equal to the average balanced gramian Σ_{av} defined in (20).

The state space transformation \tilde{T} obtained in this way can be used for model reduction of switched linear systems. After applying the transformation \tilde{T} , the new state space descriptions of the individual modes are not necessarily balanced, but are, in a sense, relatively close to being balanced. Then, the truncation decision is carried out on the basis of the eigenvalues of the product $P_{av} Q_{av}$, and a reduced-order SLS model is obtained. Of course, the question remains whether this method preserves the properties of the original SLS in the reduced order model. Indeed, the following theorem gives a sufficient condition for preserving the property of global uniform exponential stability:

Theorem VI.2: Consider the switched linear system (19) with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \dots, k$. Assume that there exists $X > 0$ such that $A_i^\top X + X A_i < 0$ for all i . Let \tilde{T} be a balancing transformation for (P_{av}, Q_{av}) with corresponding diagonal matrix $\tilde{\Sigma}$ of the form (14). Then for each positive integer $1 \leq l \leq N$ the truncated SLS of order $r = \sum_{i=1}^l m_i$ given by (21) is globally uniformly exponentially stable if

$$X P_{av} Q_{av} = Q_{av} P_{av} X. \quad (42)$$

Proof: The proof is analogous to that of Theorem V.4. By (42) we have

$$X^{\frac{1}{2}} P_{av} Q_{av} X^{-\frac{1}{2}} = X^{-\frac{1}{2}} Q_{av} P_{av} X^{\frac{1}{2}}.$$

Hence, $X^{1/2} P_{av} Q_{av} X^{-(1/2)}$ is a symmetric matrix. Thus, there exists an orthogonal matrix U such that $U X^{1/2} P_{av} Q_{av} X^{-(1/2)} U^\top = \tilde{\Sigma}^2$. Since $U X^{1/2}$ is a diagonalizing transformation for $P_{av} Q_{av}$, by Theorem III.4 \tilde{T} can be written as $\tilde{T} = \Delta^{-1} U X^{1/2}$ for some nonsingular real

matrix Δ . Applying \tilde{T} to the individual subsystems of the given SLS, the state matrices in the new coordinates are given by

$$\tilde{A}_i = \Delta^{-1} U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^\top \Delta. \quad (43)$$

By a similar argument as in the proof of Theorem V.4, we obtain

$$\tilde{A}_i^\top \Delta^\top \Delta + \Delta^\top \Delta \tilde{A}_i < 0.$$

Hence, the r th order reduced order SLS is globally uniformly exponentially stable. \blacksquare

According to Theorem VI.2, regardless of the number of modes in the original SLS, the reduced order SLS model preserves globally uniform exponential stability upon a satisfaction of the single condition (42).

The analog of the result of Theorem VI.2 holds in the case of positive real and bounded real balancing. We state the result without proof:

Corollary VI.3: Consider the switched linear system (19) with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \dots, k$. Assume that there exists $K > 0$ such that the LMI (27) [the LMI (28)] holds for all i . Define $Q_i := K_{i,\min}$ and $P_i := K_{i,\max}^{-1}$. Let \tilde{T} be a balancing transformation for (P_{av}, Q_{av}) with corresponding diagonal matrix $\tilde{\Sigma}$ of the form (14). Then for each positive integer $1 \leq l \leq N$ the truncated SLS of order $r = \sum_{i=1}^l m_i$ given by (21) is positive real (bounded real) if

$$K P_{av} Q_{av} = Q_{av} P_{av} K. \quad (44)$$

Remark VI.4: Instead of taking the average cost function (36), we can, more generally take a *weighted* average

$$f_{\text{wav}}(\tilde{T}) = \sum_{i=0}^k \alpha_i f_i(\tilde{T}) \quad (45)$$

where the α_i 's are scalars satisfying $0 \leq \alpha_i \leq 1$, and $\sum_{i=1}^k \alpha_i = 1$. Clearly, this simplifies to

$$f_{\text{wav}}(\tilde{T}) = \text{trace}[\tilde{T} \tilde{P} \tilde{T}^\top + \tilde{T}^{-\top} \tilde{Q} \tilde{T}^{-1}] \quad (46)$$

where $P_{\text{wav}} = \sum_{i=1}^k \alpha_i P_i$ and $Q_{\text{wav}} = \sum_{i=1}^k \alpha_i Q_i$. Hence, in this case, a minimizing \tilde{T} is a balancing transformation for $(P_{\text{wav}}, Q_{\text{wav}})$ where P_{wav} and Q_{wav} are corresponding weighted averages of the gramians. For example, in case that the individual modes are not of equal importance, or some information regarding the switching signal is available, an appropriate overall cost function can be chosen by manipulating the values of α_i 's. In addition it might be possible to tune the parameters α_i in such a way that the condition (42) is satisfied.

VII. NUMERICAL EXAMPLE

In this section, we will apply the results of this paper to a concrete example. We will first work out an example of a switched linear system with two modes in which simultaneous balancing is possible. Next, we will modify the SLS so that it no longer allows simultaneous balancing. Then we will apply the method

of Section VI and transform the modes of the system using a balancing transformation for the average gramians.

Consider the bimodal SLS

$$\dot{x} = A_\sigma x + B_\sigma u, \quad y = C_\sigma x \quad (47)$$

where σ is piecewise constant, taking its values in $\{1, 2\}$, and where the state equations of the modes are given by

$$\begin{aligned} A_1 &= \begin{pmatrix} -2.3333 & -3.6667 & -2.0000 \\ 3.3667 & -7.1167 & -7.8500 \\ 0.8778 & -4.1278 & -5.5500 \end{pmatrix}, \quad A_2 = A_1 + \gamma I \\ B_1 &= \begin{pmatrix} 4.1391 & -2.5590 & 1.2327 \\ 2.6502 & 3.8428 & 1.2327 \\ 1.7454 & -0.4251 & 1.2327 \end{pmatrix} \\ B_2 &= \begin{pmatrix} 4.8423 & 1.1084 & 1.5569 \\ 2.8071 & -4.5756 & 1.5569 \\ 2.5056 & -0.7863 & 1.5569 \end{pmatrix} \\ C_1 &= \begin{pmatrix} 1.2280 & 0.6693 & -1.3123 \\ -1.0617 & 1.0617 & 1.0461 \\ -0.7121 & 0.7121 & 2.1362 \end{pmatrix} \\ C_2 &= \begin{pmatrix} 0.8747 & 0.7836 & -0.3377 \\ 1.3770 & -1.3770 & -2.2192 \\ -0.8354 & 0.8354 & 2.5062 \end{pmatrix} \end{aligned}$$

where $\gamma \in \mathbb{R}$. We now distinguish between two cases.

Case 1: $\gamma = 0.75$: Our aim is to derive a reduced order model for the given SLS. Let P_1, P_2 and Q_1, Q_2 denote the reachability and observability gramians of the first and second mode, respectively. We compute

$$\begin{aligned} P_1 &= \begin{pmatrix} 4.4001 & -0.4000 & 1.9000 \\ -0.4000 & 2.0000 & -0.5000 \\ 1.9000 & -0.5000 & 1.1000 \end{pmatrix} \\ P_2 &= \begin{pmatrix} 5.4001 & -0.0001 & 2.5000 \\ -0.0001 & 2.6999 & -0.1999 \\ 2.5000 & -0.1999 & 1.6000 \end{pmatrix} \\ Q_1 &= \begin{pmatrix} 0.5222 & 0.0777 & -0.7000 \\ 0.0777 & 0.2222 & -0.2000 \\ -0.7000 & -0.2000 & 1.1999 \end{pmatrix} \\ Q_2 &= \begin{pmatrix} 0.6666 & 0.0667 & -0.9667 \\ 0.0667 & 0.3000 & -0.1333 \\ -0.9667 & -0.1333 & 1.8000 \end{pmatrix}. \end{aligned}$$

Computation shows that the following conditions hold:

$$(P_1 Q_1)(P_2 Q_2) = (P_2 Q_2)(P_1 Q_1) \quad (48)$$

$$P_1 Q_2 = P_2 Q_1. \quad (49)$$

Therefore, by Theorem IV.2, there exists a transformation T that simultaneously balances (P_1, Q_1) and (P_2, Q_2) . By computing eigenvectors, a diagonalizing transformation V for $P_1 Q_1$ and $P_2 Q_2$ is computed as

$$V = \begin{pmatrix} 0.5774 & -0.5774 & -1.7321 \\ -0.9428 & -0.4714 & 1.4143 \\ 0.8164 & -0.8166 & 0.0002 \end{pmatrix}.$$

Consequently, a simultaneous balancing transformation T can be obtained as $T = D^{-1}V$ where $D = \text{diag}(-1.7321, -1.4143, 2.4494)$ is obtained by solving the (2). The corresponding diagonal matrices obtained by applying the simultaneous balancing transformation T to (P_1, Q_1) and (P_2, Q_2) are obtained as $\Sigma_1 = \text{diag}(0.3000, 0.9000, 0.8000)$ and $\Sigma_2 = \text{diag}(0.7000, 1.1000, 0.8999)$, respectively. Hence, the average balanced gramian is computed as $\Sigma_{av} = \text{diag}(0.5000, 1.000, 0.8500)$ by (20). Note that the smallest diagonal element 0.5000 corresponds to the first state component x_1 of the SLS.

Computation shows that the state matrices A_1 and A_2 commute, and they, therefore, share a common quadratic Lyapunov function (see [15]). Hence, the given SLS is globally uniformly exponentially stable. Our intention is to compute a simultaneously balanced truncation of the SLS that preserves stability. For this, we should seek for a CQLF for A_1 and A_2 such that the constraints of Theorem V.4 are satisfied. It can be verified computationally that the positive definite matrix X given by

$$X = \begin{pmatrix} 0.1279 & 0.0388 & -0.1800 \\ 0.0388 & 0.0446 & -0.0700 \\ -0.1800 & -0.0700 & 0.2900 \end{pmatrix}$$

satisfies the constraints

$$A_1^\top X + X A_1 < 0$$

$$A_2^\top X + X A_2 < 0$$

$$X P_1 Q_1 = Q_1 P_1 X.$$

Note that we do not need to check the constraint $X P_2 Q_2 = Q_2 P_2 X$ in view of Remark V.7. Therefore, by Theorem V.4, the reduced SLS model obtained by applying the state space transformation T to the individual modes and discarding the first state component x_1 will be globally uniformly exponentially stable. The resulting second order SLS is computed as

$$\dot{\bar{x}} = \bar{A}_\sigma \bar{x} + \bar{B}_\sigma u, \quad \bar{y} = \bar{C}_\sigma \bar{x} \quad (50)$$

where

$$\begin{aligned} \bar{A}_1 &= \begin{pmatrix} -2.0001 & -0.3334 \\ -0.7501 & -3.0000 \end{pmatrix}, \quad \bar{A}_2 = \bar{A}_1 + 0.75I \\ \bar{B}_1 &= \begin{pmatrix} 1.8974 & 0.0002 & 0.0000 \\ 0.4961 & -2.1340 & -0.0000 \end{pmatrix} \\ \bar{B}_2 &= \begin{pmatrix} 1.6583 & -0.0002 & -0.0000 \\ 0.6782 & 1.8947 & -0.0000 \end{pmatrix} \\ \bar{C}_1 &= \begin{pmatrix} 1.8974 & 0.4742 \\ 0.0002 & -2.1390 \\ 0.0000 & -0.0001 \end{pmatrix} \\ \bar{C}_2 &= \begin{pmatrix} 1.6584 & 0.6279 \\ -0.0002 & 1.9118 \\ 0.0000 & -0.0000 \end{pmatrix}. \end{aligned}$$

It can be computed that $\Lambda = \text{diag}(0.25, 0.11)$ satisfies $\bar{A}_i^\top \Lambda + \Lambda \bar{A}_i < 0$ for $i = 1, 2$. Hence, as expected, global uniform exponential stability is preserved in the reduced-order SLS.

A step input is applied to the first and second input channel of the original and reduced order SLS model for two different switching signals taking values 1 or 2, randomly. For simplicity,

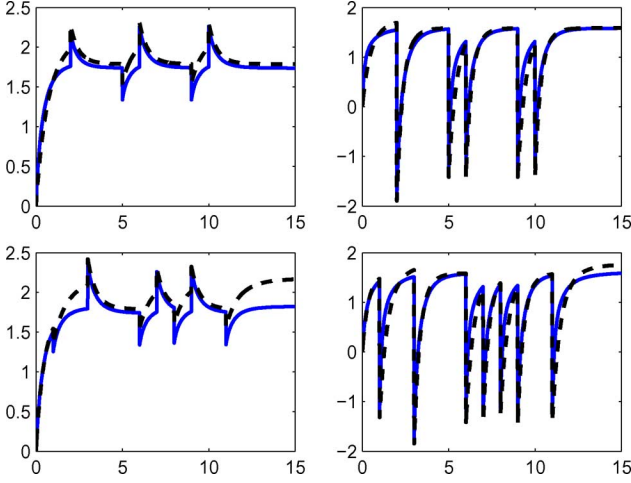


Fig. 1. Method of simultaneous balanced truncation: step responses of the original SLS (solid line) and reduced-order SLS (dotted line) from first input to first output, and second input to second output for switching signal $\sigma_1 = \{2, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1\}$ (above) and switching signal $\sigma_2 = \{1, 2, 2, 1, 1, 1, 2, 1, 2, 1, 1, 2, 2, 2, 2\}$ (below).

switching between the two subsystems takes place at integer instances of time. The corresponding outputs of the original SLS and those of the reduced order SLS model are sketched in Fig. 1.

Case 2: $\gamma = -1$: After computing the reachability and observability gramians \hat{P}_1 , \hat{P}_2 , and \hat{Q}_1 , \hat{Q}_2 of the first and second mode, it can be checked that $\hat{P}_1\hat{Q}_1$ and $\hat{P}_2\hat{Q}_2$ do not commute and, hence, a simultaneous balancing transformation does not exist. Therefore, we apply the approach developed in Section VI.

The average reachability and observability gramians P_{av} and Q_{av} are computed to be

$$P_{av} = \begin{pmatrix} 3.6658 & -0.0201 & 1.5647 \\ -0.0201 & 1.8619 & -0.0691 \\ 1.5647 & -0.0691 & 0.8741 \end{pmatrix}$$

$$Q_{av} = \begin{pmatrix} 0.4263 & 0.0122 & -0.5772 \\ 0.0122 & 0.1874 & -0.0321 \\ -0.5772 & -0.0321 & 1.0200 \end{pmatrix}.$$

A balancing transformation for (P_{av}, Q_{av}) , denoted by \tilde{T} , is computed as

$$\tilde{T} = \begin{pmatrix} 0.7175 & 0.0698 & -0.7567 \\ 0.0139 & -0.5467 & 0.1572 \\ -0.4081 & 0.1169 & 1.2491 \end{pmatrix}.$$

The corresponding diagonal matrix $\tilde{\Sigma}$ is obtained as $\tilde{\Sigma} = \text{diag}(0.7029, 0.5979, 0.3863)$. Note that the state component corresponding to the smallest diagonal element is x_3 .

As noted earlier, A_1 and A_2 commute and, therefore, share a CQLF, $x^\top X x$. To guarantee preservation of global uniform exponential stability in the reduced order model, we should look for a positive definite matrix X which satisfies the following constraints:

$$A_1^\top X + X A_1 < 0$$

$$A_2^\top X + X A_2 < 0$$

$$X P_{av} Q_{av} = Q_{av} P_{av} X.$$

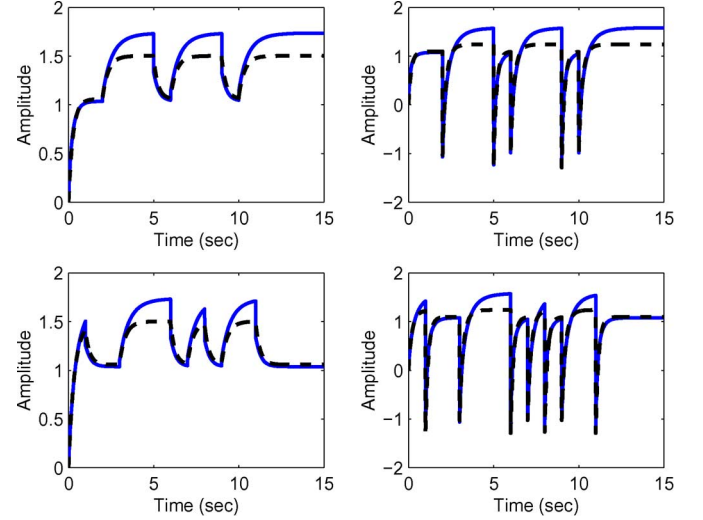


Fig. 2. Method of minimizing the overall cost function: step responses of the original SLS (solid line) and reduced-order SLS (dotted line) from first input to first output, and second input to second output for switching signal $\sigma_1 = \{2, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1\}$ (above) and switching signal $\sigma_2 = \{1, 2, 2, 1, 1, 1, 2, 1, 2, 1, 1, 2, 2, 2, 2\}$ (below).

It is easy to verify that $X = P_{av}^{-1}$ satisfies the above constraints and, hence, based on Theorem VI.2, the reduced order SLS model obtained by applying the state space transformation \tilde{T} to the individual modes and discarding the third state component x_3 is globally uniform exponentially stable. The corresponding second order SLS model is computed as

$$\dot{\bar{x}} = \bar{A}_\sigma \bar{x} + \bar{B}_\sigma u, \quad \bar{y} = \bar{C}_\sigma x \quad (51)$$

where

$$\bar{A}_1 = \begin{pmatrix} -2.8846 & -0.3463 \\ -0.6022 & -5.2693 \end{pmatrix}, \quad \bar{A}_2 = \bar{A}_1 - I$$

$$\bar{B}_1 = \begin{pmatrix} 1.8337 & -1.2462 & 0.0376 \\ -1.1172 & -2.2033 & -0.4631 \end{pmatrix}$$

$$\bar{B}_2 = \begin{pmatrix} 1.7739 & 1.0710 & 0.0475 \\ -1.0738 & 2.3934 & -0.5849 \end{pmatrix}$$

$$\bar{C}_1 = \begin{pmatrix} 1.8318 & -0.9521 \\ -1.2495 & -2.0295 \\ 0.0932 & -0.8888 \end{pmatrix}$$

and

$$\bar{C}_2 = \begin{pmatrix} 1.7657 & -1.0098 \\ 1.0607 & 2.3482 \\ 0.1095 & -1.0426 \end{pmatrix}.$$

A step input is applied to the first and second input channel of the original and reduced order SLS model for two different switching signals taking values 1 or 2, randomly. For simplicity, switching between the two subsystems takes place at integer instances of time. The corresponding outputs of the original SLS and those of the reduced-order SLS model are sketched in Fig. 2.

VIII. CONCLUSION

In this paper, a generalization of the balanced truncation scheme is investigated for model reduction of switched linear

systems. Characterizations of all balancing transformations for a single linear system are given. Clearly, making multiple linear systems balanced in general is not possible with a single state space transformation. Hence, necessary and sufficient conditions for simultaneous balancing of multiple linear systems are derived. These conditions do not depend on the particular type of balancing, and are in terms of commutativity of products of the gramians. The results obtained are applied to balanced truncation of switched linear systems. We then address the issue of preservation of stability under simultaneous balanced truncation of switched linear systems. Starting from the assumption that the original SLS has a common quadratic Lyapunov function, we establish conditions under which global uniform exponential stability of the SLS is preserved after simultaneous balanced truncation. In a similar way, the proposed conditions, with a different interpretation, are adopted for positive real and bounded real balancing. It is shown that positive realness and bounded realness of the SLS are preserved in the reduced order SLS if these conditions are satisfied.

To overcome the restrictiveness of the derived conditions, a more general balanced truncation scheme for SLS is developed based on minimizing an overall cost function. This more general approach involves balancing the average gramians rather than simultaneously balancing all the gramians corresponding to the individual modes, and, hence, the required conditions to guarantee stability, positive realness, or bounded realness are less restrictive. In case that a simultaneous balancing transformation does exist, our more general scheme reduces to the simultaneous balanced truncation scheme studied before in this paper. The proposed methods are illustrated by means of an extended numerical example.

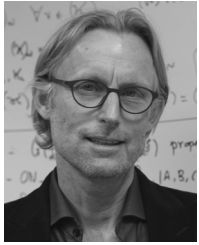
REFERENCES

- [1] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*. Philadelphia, PA: SIAM, 2005.
- [2] A. Birouche, B. Mourllion, and M. Basset, "Model order-reduction for discrete-time switched linear systems," *Int. J. Syst. Sci.*, vol. 43, no. 9, pp. 1753–1763, 2012.
- [3] U. B. Desai and D. Pal, "A transformation approach to stochastic model reduction," *IEEE Trans. Autom. Control*, vol. AC-29, no. 12, pp. 1097–1100, Dec. 1984.
- [4] G. E. Dullerud and E. G. Paganini, *A Course in Robust Control Theory: A Convex Approach*. New York: Springer, 2000.
- [5] D. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," in *Proc. 23rd IEEE Conf. Decision and Control*, Las Vegas, NV, 1984, pp. 127–132.
- [6] P. Feldman and R. W. Freund, "Efficient linear circuit analysis by Pade approximation via a Lanczos method," *IEEE Trans. Comput.-Aided Design*, vol. 14, no. 5, pp. 639–649, May 1995.
- [7] H. Gao, J. Lam, and C. Wang, "Model simplification for switched hybrid systems," *Syst. Control Lett.*, vol. 55, no. 12, pp. 1015–1021, 2006.
- [8] E. J. Grimme, "Krylov projection methods for model reduction," Ph.D. dissertation, Elect. Comput. Eng. Dept., Univ. of Illinois, Urbana-Champaign, 1997.
- [9] K. Hirata and J. P. Hespanha, " \mathcal{L}_2 -induced gain analysis for a class of switched systems," in *Proc. 48th IEEE Conf. Decision and Control*, Shanghai, China, 2009, pp. 2138–2143.
- [10] J. W. Hoffmann and P. A. Fuhrmann, "On balanced realizations of bounded real and positive real functions," *Linear Algebra App.*, vol. 245, pp. 107–146, 1996.
- [11] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York: Cambridge Univ. Press, 1988.
- [12] I. M. Jaimoukha and E. M. Kasenally, "Implicitly restarted Krylov subspace methods for stable partial realizations," *SIAM J. Matrix Anal. Applicat.*, vol. 18, pp. 633–652, 1997.
- [13] S. Lall and C. Beck, "Error-bounds for balanced model-reduction of linear time-varying systems," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 946–956, Jun. 2003.
- [14] A. Laub, M. Heath, C. Paige, and R. Ward, "Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms," *IEEE Trans. Autom. Control*, vol. AC-32, no. 2, pp. 115–122, Feb. 1987.
- [15] D. Liberzon, *Switching in Systems and Control*. Boston: Birkhauser, 2003.
- [16] C.-A. Lin and T.-Y. Chiu, "Model reduction via frequency weighted balanced realization," *Control Theory and Adv. Technol.*, vol. 8, pp. 341–351, 1992.
- [17] B. C. Moore, "Principle component analysis in linear system: Controllability, observability and model reduction," *IEEE Trans. Autom. Control*, vol. AC-26, no. 1, pp. 17–32, Feb. 1981.
- [18] C. T. Mullis and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-23, no. 9, pp. 551–562, Sep. 1976.
- [19] R. Ober, "Balanced parametrization of classes of linear systems," *SIAM J. Control Optim.*, p. 29, 1991.
- [20] P. C. Odenacker and E. A. Jonckheere, "A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds," *IEEE Trans. Circuits Syst.*, vol. CAS-35, no. 2, pp. 184–189, Feb. 1988.
- [21] L. Pernebo and L. M. Silverman, "Model reduction via balanced state space representation," *IEEE Trans. Autom. Control*, vol. AC-27, pp. 382–387, Apr. 1982.
- [22] M. Petreczky, R. Wisniewski, and J. Leth, "Theoretical analysis of balanced truncation for linear switched systems," in *Proc. 4th IFAC Conf. Anal. Design of Hybrid Syst.*, Eindhoven, The Netherlands, 2012, to be published.
- [23] J. R. Philips, L. Daniel, and L. M. Silveira, "Guaranteed passive balancing transformations for model order reduction," *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, vol. 22, no. 8, pp. 1–15, Aug. 2003.
- [24] H. Shaker and R. Wisniewski, "Generalized gramian framework for model/controller order reduction of switched systems," *Int. J. Syst. Sci.*, vol. 42, no. 8, pp. 1277–1291, 2011.
- [25] G. Wang, V. Sreeram, and W. Q. Liu, "A new frequency weighted balanced truncation method and an error bound," *IEEE Trans. Autom. Control*, vol. 44, no. 9, pp. 1734–1737, Sept. 1999.
- [26] J. T. Wen Xin Chen, "Positive realness preserving model reduction with H_∞ norm error bounds," *IEEE Trans. Circuits Syst.*, vol. 42, no. 1, pp. 23–29, Jan. 1995.
- [27] K. Zhou, "Frequency-Weighted L_∞ norm and optimal Hankel norm model reduction," *IEEE Trans. Autom. Control*, vol. 40, no. 10, pp. 1687–1699, Oct. 1995.
- [28] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Englewood Cliffs, NJ: Prentice-Hall, 1998.



Nima Monshizadeh was born in Tehran, Iran, in August 1983. He received the B.Sc. degree in electrical engineering from the University of Tehran and the M.Sc. degree in control engineering from K. N. Toosi University of Technology, Tehran. He is now pursuing the Ph.D. degree at the Systems and Control Group of the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, The Netherlands.

His research interests include model order reduction, switched systems, control configuration selection, decentralized control, networks, and multi-agent systems.



Harry L. Trentelman (M'85–SM'98) received the Ph.D. degree in mathematics from the University of Groningen, Groningen, The Netherlands, in 1985.

He is a Full Professor in Systems and Control at the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen. From 1991 to 2008, he served as an Associate Professor and later as an Adjoint Professor at the same institute. From 1985 to 1991, he was an Assistant Professor, and later an Associate Professor at the Mathematics Department, University of Technology, Eindhoven,

The Netherlands. His research interests are the behavioral approach to systems and control, robust control, model reduction, multi-dimensional linear systems, hybrid systems, analysis and control of networked systems, and the geometric theory of linear systems. He is a coauthor of the textbook *Control Theory for Linear Systems* (Springer, 2001).

Dr. Trentelman is an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and of *Systems and Control Letters*, and is past associate editor of the *SIAM Journal on Control and Optimization*.



M. Kanat Camlibel (M'05) was born in Istanbul, Turkey, in 1970. He received the B.Sc. and M.Sc. degrees in control and computer engineering from Istanbul Technical University, Istanbul, Turkey, in 1991 and 1994, respectively, and the Ph.D. degree from Tilburg University, Tilburg, The Netherlands, in 2001.

From 2001 to 2007, he held postdoctoral positions at University of Groningen and Tilburg University, assistant professor positions at Dogus University (Istanbul) and Eindhoven University of Technology. He currently holds an Assistant Professor position at University of Groningen. He is a subject editor for the *International Journal of Robust and Nonlinear Control*. His main research interests include the analysis and control of nonsmooth dynamical systems and multi-agent systems.